Vorticity and divergence or vector-invariant form

The three-component advective form of the shallow water momentum equation is

$$\frac{\partial \mathbf{A}}{\partial t} = -\left( a \frac{\partial \mathbf{A}}{\partial \mu} + b \frac{\partial \mathbf{A}}{\partial \nu} + c \frac{\partial \mathbf{A}}{\partial \lambda} \right) / R + f \mathbf{A} \times \mathbf{P} - \mathbf{P} \times \nabla \Phi$$  (1)

where \( f \) (1/s) is the Coriolis parameter, \( \Phi \) (m\(^2\)/s\(^2\)) is the fluid top geopotential, and \( k \) (m\(^2\)/s\(^2\)) is the specific kinetic energy:

$$f = 2\Omega \sin \phi; \quad (2)$$
$$\Phi = g(h + h_S); \quad (3)$$
$$k = \mathbf{A} \cdot \mathbf{A} / 2 = (a^2 + b^2 + c^2) / 2. \quad (4)$$

The time derivative, advection, metric term, Coriolis force, and pressure gradient force are all incorporated in the vector-invariant form

$$\frac{\partial \mathbf{A}}{\partial t} = (\zeta + f) \mathbf{A} \times \mathbf{P} - \mathbf{P} \times \nabla (k + \Phi) = (\zeta + f) \mathbf{S} - \mathbf{P} \times \nabla (k + \Phi). \quad (5)$$

The eastward component (Z component) of Eq. 5 reduces to

$$\frac{\partial w}{\partial t} = (\zeta + f) n - \frac{\partial (k + \Phi)}{\partial \lambda} / R \cos \phi, \quad (6)$$

and is equivalent to Eq. 19 of Williamson et al. [1992]. The time derivatives of upward relative vorticity and divergence are

$$\frac{\partial \zeta}{\partial t} = -\nabla \cdot [(\zeta + f) \mathbf{A} \times \mathbf{P}] = -\nabla \cdot [(\zeta + f) \mathbf{S}], \quad (7)$$

$$\frac{\partial (\nabla \cdot \mathbf{S})}{\partial t} = -\nabla \cdot [(\zeta + f) \mathbf{A}] - \nabla^2 (k + \Phi) = -\nabla \cdot [(\zeta + f) \mathbf{P} \times \mathbf{S}] - \nabla^2 (k + \Phi) = \mathbf{P} \cdot \nabla \times [(\zeta + f) \mathbf{S}] - \nabla^2 (k + \Phi). \quad (8)$$