Equivalence between new and old forms for ∇h

$$
\begin{aligned}
\mathbf{W} &= (-q, p, 0)/\sqrt{p^2+q^2} = (-\sin\lambda, \cos\lambda, 0) && (1)\\
\mathbf{N} &= (-rp, -qr, p^2+q^2)/\sqrt{p^2+q^2} = (-\cos\lambda\sin\phi, -\sin\lambda\sin\phi, \cos\phi) && (2)\\
\partial\lambda/\partial\delta &= \mathbf{W}\cdot\mathbf{P}\times\mathbf{U}/\cos\phi = -\sin\lambda/\cos\phi\cos\delta && (3)\\
\partial\lambda/\partial\epsilon &= \mathbf{W}\cdot\mathbf{P}\times\mathbf{V}/\cos\phi = \cos\lambda/\cos\phi\cos\epsilon && (4)\\
\partial\phi/\partial\delta &= \mathbf{N}\cdot\mathbf{P}\times\mathbf{U} = \mathbf{W}\cdot\mathbf{U} = -\cos\lambda\sin\phi/\cos\delta && (5)\\
\partial\phi/\partial\epsilon &= \mathbf{N}\cdot\mathbf{P}\times\mathbf{V} = \mathbf{W}\cdot\mathbf{V} = -\sin\lambda\sin\phi/\cos\epsilon && (6)
\end{aligned}
$$

$$
\begin{aligned}
\nabla h &= \left(\cos\delta\frac{\partial h}{\partial\delta}, \cos\epsilon\frac{\partial h}{\partial\epsilon}, \cos\phi\frac{\partial h}{\partial\phi}\right)/R =\\
&= \left[\cos\delta\left(\frac{\partial\lambda}{\partial\delta}\frac{\partial h}{\partial\lambda} + \frac{\partial\phi}{\partial\delta}\frac{\partial h}{\partial\phi}\right), \cos\epsilon\left(\frac{\partial\lambda}{\partial\epsilon}\frac{\partial h}{\partial\lambda} + \frac{\partial\phi}{\partial\epsilon}\frac{\partial h}{\partial\phi}\right), \cos\phi\frac{\partial h}{\partial\phi}\right]/R =\\
&= \left(-\sin\lambda\frac{\partial h}{\partial\lambda}/\cos\phi - \cos\lambda\sin\phi\frac{\partial h}{\partial\phi}, \cos\lambda\frac{\partial h}{\partial\lambda}/\cos\phi - \sin\lambda\sin\phi\frac{\partial h}{\partial\phi}, \cos\phi\frac{\partial h}{\partial\phi}\right)/R =\\
&= \left(\mathbf{W}\frac{\partial h}{\partial\lambda}/\cos\phi + \mathbf{N}\frac{\partial h}{\partial\phi}\right)/R = \left(\mathbf{W}\frac{\partial h}{\partial\lambda} + \mathbf{N}\cos\phi\frac{\partial h}{\partial\phi}\right)/R\cos\phi
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\partial\lambda/\partial\mu &= -pr/(p^2+q^2) = -pr/\cos^2\phi && (8)\\
\partial\lambda/\partial\nu &= -qr/(p^2+q^2) = -qr/\cos^2\phi && (9)\\
\partial\phi/\partial\mu &= q/\sqrt{p^2+q^2} = q/\cos\phi && (10)\\
\partial\phi/\partial\nu &= -p/\sqrt{p^2+q^2} = -p/\cos\phi && (11)
\end{aligned}
$$

$$
\begin{aligned}
\nabla h &= \left(r\frac{\partial h}{\partial\nu} - q\frac{\partial h}{\partial\lambda}, p\frac{\partial h}{\partial\lambda} - r\frac{\partial h}{\partial\mu}, q\frac{\partial h}{\partial\mu} - p\frac{\partial h}{\partial\nu}\right)/R = \\
&= \left[r\left(\frac{\partial\lambda}{\partial\nu}\frac{\partial h}{\partial\lambda} + \frac{\partial\phi}{\partial\nu}\frac{\partial h}{\partial\phi}\right) - q\frac{\partial h}{\partial\lambda}, p\frac{\partial h}{\partial\lambda} - r\left(\frac{\partial\lambda}{\partial\mu}\frac{\partial h}{\partial\lambda} + \frac{\partial\phi}{\partial\mu}\frac{\partial h}{\partial\phi}\right),\right. \\
&\quad \left. q\left(\frac{\partial\lambda}{\partial\mu}\frac{\partial h}{\partial\lambda} + \frac{\partial\phi}{\partial\mu}\frac{\partial h}{\partial\phi}\right) - p\left(\frac{\partial\lambda}{\partial\nu}\frac{\partial h}{\partial\lambda} + \frac{\partial\phi}{\partial\nu}\frac{\partial h}{\partial\phi}\right)\right]/R = \\
&= \left(-rqr\frac{\partial h}{\partial\lambda}/\cos^2\phi - rp\frac{\partial h}{\partial\phi}/\cos\phi - q\frac{\partial h}{\partial\lambda}, p\frac{\partial h}{\partial\lambda} + rpr\frac{\partial h}{\partial\lambda}/\cos^2\phi - rq\frac{\partial h}{\partial\phi}/\cos\phi,\right. \qquad (12)\\
&\quad \left. -qpr\frac{\partial h}{\partial\lambda}/\cos^2\phi + qq\frac{\partial h}{\partial\phi}/\cos\phi + pqr\frac{\partial h}{\partial\lambda}/\cos^2\phi + pp\frac{\partial h}{\partial\phi}/\cos\phi\right)/R = \\
&= \left(-q\frac{\partial h}{\partial\lambda}/\cos^2\phi - rp\frac{\partial h}{\partial\phi}/\cos\phi, p\frac{\partial h}{\partial\lambda}/\cos^2\phi - rq\frac{\partial h}{\partial\phi}/\cos\phi, \cos\phi\frac{\partial h}{\partial\phi}\right)/R = \\
&= \left(\mathbf{W}\frac{\partial h}{\partial\lambda}/\cos\phi + \mathbf{N}\frac{\partial h}{\partial\phi}\right)/R = \left(\mathbf{W}\frac{\partial h}{\partial\lambda} + \mathbf{N}\cos\phi\frac{\partial h}{\partial\phi}\right)/R\cos\phi
\end{aligned}
$$

$$\mathbf{P} \times \nabla h = \mathbf{P} \times \left(r\frac{\partial h}{\partial \nu} - q\frac{\partial h}{\partial \lambda}, p\frac{\partial h}{\partial \lambda} - r\frac{\partial h}{\partial \mu}, q\frac{\partial h}{\partial \mu} - p\frac{\partial h}{\partial \nu} \right) /R \tag{13}$$

The third component of $\mathbf{P} \times \nabla h$ is:

$$\begin{aligned}
&\left[p\left(p\frac{\partial h}{\partial \lambda} - r\frac{\partial h}{\partial \mu} \right) - q\left(r\frac{\partial h}{\partial \nu} - q\frac{\partial h}{\partial \lambda} \right) \right] /R = \\
&= \left[(p^2+q^2)\frac{\partial h}{\partial \lambda} - pr\left(\frac{\partial \lambda}{\partial \mu}\frac{\partial h}{\partial \lambda} + \frac{\partial \phi}{\partial \mu}\frac{\partial h}{\partial \phi} \right) - qr\left(\frac{\partial \lambda}{\partial \nu}\frac{\partial h}{\partial \lambda} + \frac{\partial \phi}{\partial \nu}\frac{\partial h}{\partial \phi} \right) \right] /R = \\
&= \left\{ (p^2+q^2)\frac{\partial h}{\partial \lambda} - pr\left[-pr\frac{\partial h}{\partial \lambda}/(p^2+q^2) + q\frac{\partial h}{\partial \phi}/\sqrt{p^2+q^2} \right] \right. \\
&\quad \left. -qr\left[-qr\frac{\partial h}{\partial \lambda}/(p^2+q^2) - p\frac{\partial h}{\partial \phi}/\sqrt{p^2+q^2} \right) \right\} /R = \\
&= \left[(p^2+q^2)\frac{\partial h}{\partial \lambda} + prpr\frac{\partial h}{\partial \lambda}/(p^2+q^2) + qrqr\frac{\partial h}{\partial \lambda}/(p^2+q^2) \right] /R = \\
&= \left[(p^2+q^2)\frac{\partial h}{\partial \lambda} + r^2\frac{\partial h}{\partial \lambda} \right] /R = \frac{\partial h}{\partial \lambda}/R
\end{aligned} \tag{14}$$

Thus:

$$\mathbf{P} \times \nabla h = \left(\frac{\partial h}{\partial \mu}, \frac{\partial h}{\partial \nu}, \frac{\partial h}{\partial \lambda} \right) /R. \tag{15}$$