Compute relative vorticity using Green's Theorem

$$\zeta = -\left[\frac{\partial(a\cos\delta)}{\partial\delta} + \frac{\partial(b\cos\epsilon)}{\partial\epsilon} + \frac{\partial(c\cos\phi)}{\partial\phi}\right]/R. \tag{1}$$

The unit vector in the direction of δ is $\nabla\delta/|\nabla\delta| = \mathbf{P}\times\mathbf{U}$. For a grid cell on Face 1 of the CS grid, the outgoing perpendicular unit vectors for the right and left edges are $\mathbf{V}$ and $-\mathbf{V}$, and for the top and bottom edges are $-\mathbf{U}$ and $\mathbf{U}$. By Green's Theorem, the integral of $\partial(a\cos\delta)/\partial(R\delta)$ over the area of the grid cell equals the line integral of $(a\cos\delta)$ multiplied by $\mathbf{P}\times\mathbf{U}$ dotted with normal edge unit vector. Thus, the summation of all terms for the right edge line integral is

$$\begin{aligned} &L(\mathbf{P}\times\mathbf{U}\cdot\mathbf{V}a\cos\delta + \mathbf{P}\times\mathbf{V}\cdot\mathbf{V}b\cos\epsilon + \mathbf{P}\times\mathbf{W}\cdot\mathbf{V}c\cos\phi) = \\ &= L(\mathbf{P}\times\mathbf{U}\cdot\mathbf{V}a\cos\delta + \mathbf{P}\times\mathbf{W}\cdot\mathbf{V}c\cos\phi) = L(a\cos\nu - c\sin\nu) = L\mathbf{A}\cdot\mathbf{V} \end{aligned} \tag{2}$$

where L is the arc length of the edge. For the bottom edge

$$\begin{aligned} &L(\mathbf{P}\times\mathbf{U}\cdot\mathbf{U}a\cos\delta + \mathbf{P}\times\mathbf{V}\cdot\mathbf{U}b\cos\epsilon + \mathbf{P}\times\mathbf{W}\cdot\mathbf{U}c\cos\phi) = \\ &= L(\mathbf{P}\times\mathbf{V}\cdot\mathbf{U}b\cos\epsilon + \mathbf{P}\times\mathbf{W}\cdot\mathbf{U}c\cos\phi) = L(-b\sin\mu + c\cos\mu) = L\mathbf{A}\cdot\mathbf{U}. \end{aligned} \tag{3}$$

$$\zeta = -[(L\mathbf{A}\cdot\mathbf{V})_{RIGHT} - (L\mathbf{A}\cdot\mathbf{V})_{LEFT} - (L\mathbf{A}\cdot\mathbf{U})_{TOP} + (L\mathbf{A}\cdot\mathbf{U})_{BOTTOM}]/K \tag{4}$$

where K is the grid cell area. If $\mathbf{A}$, $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ and $\mathbf{S}$ are all perpendicular to $\mathbf{P}$, then $\mathbf{A}\cdot\mathbf{V} = \mathbf{P}\times\mathbf{A}\cdot\mathbf{P}\times\mathbf{V} = -\mathbf{A}\times\mathbf{P}\cdot\mathbf{P}\times\mathbf{V} = -\mathbf{S}\cdot\mathbf{P}\times\mathbf{V}$, and

$$\zeta = [(L\mathbf{S}\cdot\mathbf{P}\times\mathbf{V})_{RIGHT} - (L\mathbf{S}\cdot\mathbf{P}\times\mathbf{V})_{LEFT} - (L\mathbf{S}\cdot\mathbf{P}\times\mathbf{U})_{TOP} + (L\mathbf{S}\cdot\mathbf{P}\times\mathbf{U})_{BOTTOM}]/K \tag{5}$$

which is a well known formula for the computation of ζ using Green's Theorem. A similar formula is valid for spherical polygons, not just quadrilaterals.