Vorticity and divergence or vector-invariant form

The three-component advective form of the shallow water momentum equation is

$$\frac{\partial \mathbf{A}}{\partial t} = -\left(a\frac{\partial \mathbf{A}}{\partial \mu} + b\frac{\partial \mathbf{A}}{\partial \nu} + c\frac{\partial \mathbf{A}}{\partial \lambda}\right)/R + f\mathbf{A}\times\mathbf{P} - \mathbf{P}\times\nabla\Phi \tag{1}$$

where f (1/s) is the Coriolis parameter, Φ ($\mathrm{m}^2/\mathrm{s}^2$) is the fluid top geopotential, and k ($\mathrm{m}^2/\mathrm{s}^2$) is the specific kinetic energy:

$$f = 2\Omega\sin\phi; \tag{2}$$

$$\Phi = g(h + h_S); \tag{3}$$

$$k = \mathbf{A}\cdot\mathbf{A}/2 = (a^2 + b^2 + c^2)/2. \tag{4}$$

The time derivative, advection, metric term, Coriolis force, and pressure gradient force are all incorporated in the vector-invariant form

$$\frac{\partial \mathbf{A}}{\partial t} = (\zeta + f)\mathbf{A}\times\mathbf{P} - \mathbf{P}\times\nabla(k+\Phi) = (\zeta + f)\mathbf{S} - \mathbf{P}\times\nabla(k+\Phi). \tag{5}$$

The eastward component ($\mathbf{Z}$ component) of Eq. D5 reduces to

$$\frac{\partial w}{\partial t} = (\zeta + f)n - \frac{\partial(k+\Phi)}{\partial\lambda}/R\cos\phi, \tag{6}$$

and is equivalent to Eq. 19 of Williamson et al. [1992]. The time derivatives of upward relative vorticity and divergence are

$$\frac{\partial \zeta}{\partial t} = -\nabla\cdot[(\zeta+f)\mathbf{A}\times\mathbf{P}] = -\nabla\cdot[(\zeta+f)\mathbf{S}], \tag{7}$$

$$\begin{aligned}\frac{\partial(\nabla\cdot\mathbf{S})}{\partial t} &= -\nabla\cdot[(\zeta+f)\mathbf{A}] - \nabla^2(k+\Phi) = -\nabla\cdot[(\zeta+f)\mathbf{P}\times\mathbf{S}] - \nabla^2(k+\Phi) = \\ &= \mathbf{P}\cdot\nabla\times[(\zeta+f)\mathbf{S}] - \nabla^2(k+\Phi).\end{aligned} \tag{8}$$