Comparison of 3 symmetric versus 2 non-orthogonal velocity components

A single layer cubed-sphere grid is tested with four different horizontal resolutions: 22x22x6, 44x44x6, 88x88x6, and 176x176x6. The initial height field $h = 10000$ (m) is uniform everywhere; $h_S = 0$. An axis pole on the unit sphere is chosen as $\mathbf{I} = (1, 2, 3)/\sqrt{14}$. The angular rotation coordinate from $-\mathbf{I}$ to $\mathbf{I}$ is η, the axis' latitude. Orthogonal unit vectors at point $\mathbf{P}$ with respect to this axis are $\mathbf{F} = \mathbf{I} \times \mathbf{P}/|\mathbf{I} \times \mathbf{P}|$ and $\mathbf{E} = \mathbf{P} \times \mathbf{F}$. If $\mathbf{I}$ were the Earth's North Pole, then $\mathbf{F}$ would equal $\mathbf{W}$ and $\mathbf{E}$ would equal $\mathbf{N}$. The initial velocity field is $\mathbf{S}$ (m/s) $= 50\mathbf{F}\cos\eta + 10\mathbf{E}\cos(9\eta)$; it shows significant variability for the coarsest resolution. It was the first velocity field programmed and was not chosen to improve nor diminish the following tests. $\mathbf{S}$ is horizontal because $\mathbf{F}$ and $\mathbf{E}$ are horizontal. Both primary cell mean and edge values of $\mathbf{S}$ are computed.

The initial advection terms contributing to $\partial\mathbf{S}/\partial t$, which includes the metric term but not the Coriolis force, are differentiated from the trigonometric form of $\mathbf{S}$. The global mean value of $|\partial\mathbf{S}/\partial t|$ times 10^6 is shown in Table 1. For each resolution, the advective terms only are integrated producing changes for Δu, Δv and Δw of primary cells by a single forward step with Δt of one thousandth of the normal time step (so that the linear upstream advection is less important). Flux form integration is performed using two techniques: in (am), primary cell mean angular momentum changes are summed from edge flux computations as described in Section 7; and in (mt), velocity changes are summed from edge fluxes of velocity followed by addition of the metric term using the initial specified primary cell mean velocity components.

The following tests compare two non-orthogonal velocity components (2com) versus three symmetric non-independent velocity components (3com). For test (2com) and each face of the cube, two components are chosen that are perpendicular to grid edges on that face and not the velocity component whose pole is centered at the face (see Figure 1). For each face, horizontal $\Delta\mathbf{S}$ is resolved by the two chosen non-orthogonal components. The area weighted global root mean square difference of $|\Delta\mathbf{S}/\Delta t - \partial\mathbf{S}/\partial t|$ times 10^6 is shown in Table 1 for each integration technique. Advection of velocity with metric term added (2com,mt) is superior to advection of angular momentum (2com,am).

For test (3com), although $\partial\mathbf{S}/\partial t$ is horizontal, $\Delta\mathbf{S}$ may not be since it is determined by all three components including a component whose pole may touch a grid cell corner. The global value of $|\Delta\mathbf{S}/\Delta t - \partial\mathbf{S}/\partial t|$ is shown in Table 1 for each integration technique. Using the alignment algorithm of Section 5, $\Delta\mathbf{S}_{NEW}$ is computed from $\Delta\mathbf{S}$ and $|\Delta\mathbf{S}_{NEW}/\Delta t - \partial\mathbf{S}/\partial t|$ labeled (a) is shown in Table 1. $\Delta\mathbf{S}_{NEW}$ is horizontal. Advection of three component angular momentum followed by component alignment is most accurate (3com,am,a).

The final row in Table 1 shows the benefit in performing computations with three directions. It is one minus the ratio of row (3com,am,a) divided by row (2com,mt), converted to percent. The benefit occurs for other choices of $\mathbf{I}$ and $\mathbf{S}$; it depends somewhat on the location of $\mathbf{I}$, but more strongly on the form chosen for $\mathbf{S}$. Although the global average error is less with three components, this is not the case for many grid cells because of the oscillating function chosen.

Table 1: Row 1 shows global magnitude $|\partial\mathbf{S}/\partial t|$ from differentiation times 10^6. Rows 2 or 3 show $|\Delta\mathbf{S}/\Delta t - \partial\mathbf{S}/\partial t|$ using only two components, advecting angular momentum or velocity and the metric term. Rows 4 or 5 show $|\Delta\mathbf{S}/\Delta t - \partial\mathbf{S}/\partial t|$ using three components with the two integration techniques. Rows 6 or 7 show $|\Delta\mathbf{S}_{NEW}/\Delta t - \partial\mathbf{S}/\partial t|$ after alignment. Row 8 shows the benefit from using three directions as opposed to only two; it is one minus the ratio of row 6 divided by row 3.

	22x22	44x44	88x88	176x176
$\lvert\partial\mathbf{S}/\partial t\rvert$	164.90	164.90	164.90	164.90
2com,am	4.7280	1.1967	.3001	.0751
2com,mt	4.7161	1.1936	.2993	.0749
3com,am	4.7760	1.2088	.3031	.0758
3com,mt	5.5296	1.8131	.7338	.3415
3com,am,a	4.6201	1.1697	.2934	.0734
3com,mt,a	4.6851	1.1865	.2977	.0745
Benefit	2.04%	2.00%	1.99%	1.99%

Flux information at an edge is not exact, being the product of single point values as opposed to an integral of the product over an arc edge. This causes Δu, Δv and Δw to contain minor errors. If each component had an error of 1 (m/s) and $\Delta\mathbf{S}$ is resolved by just two components, then the magnitude error in $\Delta\mathbf{S}$ is $\sqrt{2}$ if the components are orthogonal, but if component unit vectors are separated by a 120° angle, which may occur at face corners, the maximum error is 2. This explains why orthogonal components are preferred. Further, discontinuity while switching velocity components at face edges in two-component model may cause additional error, but it is ignored by these tests. The maximum error in $\Delta\mathbf{S}_{NEW}$ with three components at face corners, after alignment, is 4/3, less than $\sqrt{2}$ because the components are maximally separated. A component, having a pole in a face, may still improve computations on the face, although it will have less importance near its pole.